Numerical Solution of ODEs of 1st order:

In this chapter, we consider some of the numerical methods to solve 1st order ODEs. Generally, the value of the dependent variable \( y \) will be given at the initial value of \( x \) of the order of \( y \) of the ODE \( \frac{dy}{dx} = f(x,y) \).

Such a DE is referred to as an Initial Value Problem (IVP).

Here, we study first Picard’s & Taylor’s series methods.

These methods give the solution in the form of a series, so they are called series solutions for these DEs.

1. Picard’s method of successive approximations:

Consider the DE of 1st order, \( \frac{dy}{dx} = f(x,y) \), given \( y(x_0) = y_0 \).

\( y_0 \) can be written as \( y_0 = f(x_0, y_0) \)

On integrating the LHS between the limits \( y_0 \) to \( y \), and the RHS between the limits \( x = x_0 \) to \( x \), we get

\[
\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y) \, dx \Rightarrow y - y_0 = \int_{x_0}^{x} f(x,y) \, dx.
\]

\( \Rightarrow y = y_0 + \int_{x_0}^{x} f(x,y) \, dx. \)

The first approximation \( y_1 \) in this method is obtained by taking \( y = y_0 \) on the RHS, and writing the corresponding \( y \) on LHS as \( y_1 \), \( y_1 = y_0 + \int_{x_0}^{x} f(x,y_0) \, dx. \)

By taking \( y = y_1 \) on RHS, we obtain \( y_2 \) on LHS as \( y_2 = y_0 + \int_{x_0}^{x} f(x,y_1) \, dx. \).
Continuing like this we get:
\[ f_{m+1} = y_0 + \int_{x_0}^{x} f(x, y_m) \, dx. \]

This is the iterative formula used in the Picard's method to obtain the successive approximations.

Problem: Using Picard's method, find a solution up to 5th approximation of the DE \( \frac{dy}{dx} = 4y + x ; \ y(0) = 1 \). Verify the answer.

\[ y_1 = y_0 + \int_{x_0}^{x} f(x, y_0) \, dx = 1 + \int_{0}^{x} (4x + x^2) \, dx = 1 + x + x^2 - \frac{x^3}{3}. \]

\[ y_2 = y_0 + \int_{x_0}^{x} f(x, y_1) \, dx = 1 + \int_{0}^{x} \left( \frac{(1+x+x^2)}{2} + x \right) \, dx = 1 + x + x^2 + \frac{x^3}{6}. \]

\[ y_3 = y_0 + \int_{x_0}^{x} f(x, y_2) \, dx = 1 + \int_{0}^{x} \left( \frac{(1+x+x^2)}{2} + x \right) \, dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12}. \]

\[ y_4 = y_0 + \int_{x_0}^{x} f(x, y_3) \, dx = 1 + \int_{0}^{x} \left( \frac{(1+x+x^2)}{2} + x \right) \, dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{720}. \]

\[ y_5 = y_0 + \int_{x_0}^{x} f(x, y_4) \, dx = 1 + \int_{0}^{x} \left( \frac{(1+x+x^2)}{2} + x \right) \, dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{720} + \frac{x^6}{540}. \]

Thus \( y_5 \) is the 5th approximation.

Also, \( \frac{dy}{dx} = y = x \) is the heiknitze linear in \( x \).

\[ IF = \int_{y}^{1} y \, dy = e^x - e. \]

So the solution is \( y \cdot IF = \int_{y}^{1} y \, dy \)

\[ y \cdot e^{-x} = \int_{e^{-x}}^{e^x} e^{-x} \, dx \Rightarrow ye^{-x} = x(e^{-x}) - \int(e^x) \, dx \]

\[ ye^{-x} = -e^{-x} - e^x + c. \]

\[ y = ce^x - x - 1 \]

\[ y|_{x=0} = ce^{0} - 0 - 1 = c - 1 = 1 \]

\[ c - 1 = 1 \Rightarrow c = 2. \]
Use Picard's method to obtain the third approximations to
the solution of \( \frac{dy}{dx} + y = e^x \), \( y(0) = 1 \) and hence find \( y(0.2) \).

Find also the true value of the solution of the given eqn.

Sohn: The formula is
\( y = 1 + \int_0^y (e^y - y) \, dx \)

\( \implies y_1 = e^x - x \)

\( y_2 = 1 + x^2 \)

\( y_3 = e^x - x - \frac{x^3}{6} \) is the required sohn

\( \implies y(0.2) \approx 1.0200694 \).

The exact sohn is
\( y = e^x \int e^x - e^x \, dx + c \implies y = e^x + Ce^{-x} \); \( C = \frac{1}{2} \)

so \( y = \cosh x \) so \( y \bigg|_{x=0.2} \approx 1.0200668 \).

3. Employ Picard's method to find the sohn of the DE
\( \frac{dy}{dx} = x^2 + y^2 \), given that \( y = 0 \) when \( x = 0 \). Hence find \( y(0.1) \) correct
to 4 decimal places.

Sohn: The formula is
\( y = \int_0^x (x^2 + y^2) \, dx \implies y_1 = \frac{2}{3}; y_2 = \frac{2^3 + x^7}{6} \)

\( \implies y(0.1) = 0.00033 \)

4. Given \( \frac{dy}{dx} = x \cdot e^y \), \( y(0) = 0 \) determine \( y(0.1), y(0.2) \)
and \( y(1) \) using Picard's method. Compare the sohn with

the exact solution

Sohn: \( y = e \); \( y(0.1) = 0.1055; y(0.2) = 0.0202 \)

\( y(1) = 0.6487 \).

5. Solve \( y' = 1 + 2x \cdot y, \ y(0) = 0 \) by Picard's method.

Sohn: \( y = x + 2x^2 \frac{x}{3} + 4x^5 \frac{x}{15} \).
\[ \frac{dy}{dx} = x - y \quad y(0) = 1, \quad \text{find } y(0.1), y(0.2) \text{ using Picard's method.} \]

**Solution:** \[ y = 1 - x + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{120}; \quad y(0) = 1 \]

**Note:** This Picard's method of successive approximations cannot be applied to find a solution of every 1st order DE. For some of them it is difficult to obtain a series form as the integrand may not be integrable. Therefore, some more methods are available to solve this type of DEs. Let us demonstrate this limitation with the following example.

(1) Find the value of \( \frac{dy}{dx} = \frac{y - x}{y + x} \); \( y(0) = 1 \) using Picard's method at \( x = 0.1 \) and \( x = 0.2 \).

**Solution:** By Picard's method, \[ y = y_0 + \int_0^x \left( \frac{y - x}{y + x} \right) \, dx \]

\[ = 1 + \int_0^x \frac{1 - x}{1 + x} \, dx \times \]

\[ = 1 + \int_0^x \left( -1 + \frac{2}{1 + x} \right) \, dx \]

\[ = 1 + \left( -x + 2 \log (1 + x) \right) \biggr|_0^x \]

\[ = 1 + \left( -x + 2 \log (1 + x) \right) \]

Since further integration is not possible, further approximations are not possible. So we may not get a solution to the required accuracy.
Taylor's Series Method:

To solve a first order differential equation \( \frac{dy}{dx} = f(x, y) \) by this method, we consider the Taylor's series expansion of \( y(x) \) w.r.t. the point \( (x_0) \), i.e., in powers of \( (x-x_0) \):

\[ y(x) = y(x_0) + (x-x_0) \frac{y'(x_0)}{1!} + (x-x_0)^2 \frac{y''(x_0)}{2!} + \ldots \]

Problems: Use Taylor's method to find approximate value of \( y(1.1) \) and \( y(1.2) \) for the DE \( y' = xy^{1/3} \), \( y(1) = 1 \). Compare the numerical solution obtained with exact solution.

Sols: Given \( f(x, y) = xy^{1/3} = y' \), \( y' \mid_{x_0 = 1} = 1 \)

\[ \begin{align*}
  y'' &= x \cdot \frac{1}{3} y^{-2/3} y' + y'' \mid_{x_0 = 1} = \frac{x}{3} y^{-1/3} + y^{1/3} \\
  y''' &= \frac{2}{3} x \cdot \left(-\frac{1}{3} y^{-4/3}\right) y' + \frac{2x}{3} y^{-2/3} y'^2 + \frac{y}{3} y^{-1/3} y' \mid_{x_0 = 1} = \\
  &= -\frac{x}{9} y^{-4/3} + \frac{2x}{3} y^{-2/3} + \frac{y}{3} y^{1/3} \\
  y''' \mid_{x_0} &= \frac{1}{9} + \frac{2}{3} + \frac{1}{3} + \frac{8}{9} = \frac{5}{3}
\end{align*} \]

Now, substituting in Taylor's series, we get:

\[ y(x) = y(x_0) + (x-x_0) \frac{y'(x_0)}{1!} + (x-x_0)^2 \frac{y''(x_0)}{2!} + \ldots \]

Taking \( x = 1.1 \):

\[ y(1.1) = 1 + (0.1)(1) + (0.1)^2 \frac{1}{2!} - \frac{1}{9} + (0.1)^3 \frac{8}{9} = 1.1067 \]

Taking \( x = 1.2 \):

\[ y(1.2) = y(1.1) + (1.2-1.1) \frac{y'(1.1)}{1!} + (1.2-1.1)^2 \frac{y''(1.1)}{2!} + \ldots \]

\[ y(1.2) = 1.1067 + (1.2-1.1) \frac{1}{2!} + \ldots \]
\[
\begin{align*}
\gamma_1 &= \alpha_1 \gamma_1^{1/3} = (1.1) (1.1067)^{1/3} = 1.13782 \\
\gamma_1'' &= \frac{1}{3} \gamma_1^{2/3} = \frac{1}{3} (1.1)^2 (1.1067)^{2/3} \\
&= 1.4243 \\
\gamma_1''' &= 0.9297.
\end{align*}
\]

On substitution in Taylor series, we get
\[
\gamma_2 = 1.1067 + 0.113782 + 0.00712 + 0.00015475 -
\]
\[= 1.2278.\]

So \( y_3 = \gamma_1^{1/3} \) is given by \( 1.3639 \).

The analytical solution is: \( \frac{dy}{\gamma_3^{1/3}} = x \, dx \) on separating variables

\[
\frac{3}{2} \, \gamma_3^{1/3} = \frac{x^2}{2} + c
\]

\( \frac{3}{2} (1)^{1/3} = \frac{1}{2} + c \) \( \Rightarrow \frac{3}{2} = \frac{1}{2} + c \) \( \Rightarrow c = 1. \)

\( \therefore \) The solution is \( \frac{3}{2} \, \gamma_3^{1/3} = \frac{x^2}{2} + 1 \)

or \( \gamma_3^{1/3} = \frac{x^2 + 2}{3} \)

So \( y_1 (1.1) = \left[ \frac{(1.1)^2 + 2}{3} \right]^{3/2} \Rightarrow y_1 (1.1) = 1.1068 \)

\( y_1 (1.2) = \left[ \frac{(1.2)^2 + 2}{3} \right]^{3/2} = 1.1467 \) \( \Rightarrow y_1 (1.2) = 1.2278 \)

\( y_1 (1.3) = \left[ \frac{(1.3)^2 + 2}{3} \right]^{3/2} = 1.283 \) \( \Rightarrow y_1 (1.3) = 1.364 \)
2. Use Taylor's series method to find the approximate value of $y$ when $x = 0.1$ given $y(0) = 1$ and $y' = 3x + y^2$.

**Solution:**

$y' = 3x + y^2 \Rightarrow y'_0 = 3x_0 + y^2_0 = 1$

$\Rightarrow y'' = 3 + 2yy' \Rightarrow y''_0 = 3 + 2y_0y'_0 = 5$

$\Rightarrow y''' = 2yy'' + 2(y')^2 \Rightarrow y'''_0 = 2y_0y''_0 + 2(y'_0)^2 = 12$

$\Rightarrow y^{(4)} = 2yy''' + 2y'y'' + 4yy'' + 4y'y'' \Rightarrow y^{(4)}_0 = 54$

Given $x_0 = 0; \quad y_0 = 1; \quad h = 0.1$

$y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \cdots$

$= 1 + (0.1) 1 + (0.1)^2 \cdot 5 + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{24} + \cdots$

$= 1.127$

3. Find by Taylor's series method the value of $y$ at $x = 0.1$ to five places of decimal from $y' = x^2y - 1; \quad y(0) = 1$.

**Solution:**

$y' = x^2y - 1 \Rightarrow y'_0 = x_0^2y_0 - 1 = -1$

$y'' = 2xy + x^2y' \Rightarrow y''_0 = 2x_0y_0 + x_0^2y'_0 = 0$

$y''' = 2xy' + 2y + x^2y'' + 2xy' \Rightarrow y'''_0 = 2$

$y^{(4)} = 2xy'' + 2y' + 2y'y'' + x^2y''' + 2xy'' + 2xy'' + 2y' = -6$

$y_1 = y(0.1) = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \frac{h^4}{4!} y^{(4)}_0 + \cdots$

$= -1 + (0.1)(-1) + (0.1)^2 \cdot 0 + (0.1)^3 \cdot \frac{2}{6} + (0.1)^4 \cdot \frac{(-6)}{24} + \cdots$

$= 0.9003$
4. Solve \( \frac{dy}{dx} = xy + 1 \) and \( y(0) = 1 \) using Taylor's Series method and compute \( y(0.1) \).

Solution: \( x_0 = 0; y_0 = 1; h = 0.1 \).

\[
\begin{align*}
y' &= xy + 1; \quad y'_0 = 1 \\
y'' &= xy' + y; \quad y''_0 = 1 \\
y''' &= xy'' + y' + y'; \quad y'''_0 = 2 \\
y'''' &= xy''' + y'' + 2y''; \quad y''''_0 = 3 \\
y(0.1) &= y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0'''' + \ldots \\
&= 1.1053.
\end{align*}
\]

5. Solve the following first order differential equations using Taylor's Series method:

i) \( y' = xy^2 - 1 \), \( y(0) = 1 \). Compute \( y(0.3) \). \( y(0.3) = 0.97 \)

ii) \( y' = y - x \), \( y(0) = 1 \) in \( 0 \leq x \leq 0.2 \) up to 3rd approximation

\[
\text{Solution: } 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{60};
\]

\( y(0.2) = 1.018 \)

iii) \( y' = x + y^2 + 1 \), \( y(0) = 0 \). Obtain the series approximation up to the fifth degree terms.

\[
\text{Solution: } x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{20} + \frac{x^5}{120};
\]

iv) Solve \( y' = x - y \), \( y(0) = 1 \) using Taylor's Series method and compute \( y(0.1) \) and \( y(0.2) \).

\[
\text{Solution: } y = c_0 + \frac{h}{1!} c_0' + \frac{h^2}{2!} c_0'' + \frac{h^3}{3!} c_0''' + \frac{h^4}{4!} c_0'''' + \ldots.
\]

\( y(0.1) = 0.91381; \quad y(0.2) = 0.8512 \).
Modified Euler's method

This method is an enhancement of Euler's method. Modified Euler's method, we take the average of the slopes at \((x_0, y_0)\) & \((x_1, y_1)\) for two points where as in the Euler's method, the slope is considered at only one point.

The formula of Euler's method is \( y_{n+1} = y_n + hf(x_n, y_n) \)

Iterative formula for the Modified Euler's method is

\[
y^{(n+1)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1) \right]
\]

**Problem 1:** Using modified Euler's method find \( y(0.2) \)
and \( y(0.4) \) given \( y' = ye^x \), \( y(0) = 0 \).

**Solu:** \( x_0 = 0; y_0 = 0; h = 0.2 \)

By Euler's formula, \( y_1^{(0)} = y_0 + hf(x_0, y_0) = 0 + 0.2(y_0 + e^{x_0}) = 0 + 0.2(0 + e^0) = 0.2 \)

Now \( x_1 = 0.2 \) and \( f(x_1, y_1^{(0)}) = f(0.2, 0.2) = 0.2 + e^{0.2} = 1.4214 \)

\[
y_1^{(1)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] = 0 + 0.1 \left[ 0 + e^0 + 0.2 + e^{0.2} \right] = 0.24214
\]

\[
y_1^{(2)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(1)}) \right] = 0 + 0.1 \left[ 0 + e^0 + 0.24214 + e^{0.2} \right] = 0.2463
\]
\[ y_1^{(3)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(2)}) \right] = 0 + \frac{0.2}{2} \left[ 1 + 0.2463 + e^{0.2} \right] = 0.2468 \]

\[ y_1^{(4)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(3)}) \right] = 0 + \frac{0.2}{2} \left[ 1 + 0.2463 + e^{0.2} \right] = 0.2468 \]

**Problem 2:** Given \( \frac{dy}{dx} = -xy \), \( y(0) = 2 \). Compute \( y(0.2) \) in steps of 0.1 using modified Euler's method.

**Solution:** \( x_0 = 0; \ y_0 = 2; \ h = 0.2 \)

By Euler's formula, \( y_1 = y_0 + h f(x_0, y_0) \)

Take it as \( y_1^{(1)} \),

\[ y_1^{(1)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(1)}) \right] \]

\[ y_1^{(1)} = y_0 + \frac{1}{2} \left[ -x_0 y_0^2 + (-x_1 y_1^{(1)}) \right] = 2 + \frac{0.1}{2} \left[ -x_0 y_0^2 + (-x_1 y_1^{(1)}) \right] \]

\[ y_1^{(1)} = 1.98 \]

\[ y_1^{(2)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(2)}) \right] \]

\[ y_1^{(2)} = y_0 + \frac{1}{2} \left[ -x_0 y_0^2 + (-x_1 y_1^{(2)}) \right] = 1.9804 \]

\[ y_1^{(3)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(3)}) \right] \]

\[ y_1^{(3)} = y_0 + \frac{1}{2} \left[ -x_0 y_0^2 + (-x_1 y_1^{(3)}) \right] = 1.9804 \]

\[ y_1^{(4)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(4)}) \right] \]

\[ y_1^{(4)} = y_0 + \frac{1}{2} \left[ -x_0 y_0^2 + (-x_1 y_1^{(4)}) \right] = 1.9804 \]

\[ x_1 = 0.1; \ y_1^{(2)} = 1.9804; \ x_2 = 0.2 \& h = 0.1 \]

\[ y_2^{(2)} = y_1^{(2)} + hf(x_1, y_1^{(2)}) = 1.9804 + (0.1) (-0.1)(1.9804^2) = 1.94118 \]

\[ y_2^{(3)} = y_1^{(3)} + hf(x_1, y_1^{(3)}) = 1.9804 + (0.1) (-0.1)(1.9804^2) = 1.94118 \]
\[ y_2^{(1)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_2, y_2^{(0)}) \right] = 1.9804 + 0.1 \left[ -0.3922 + (-0.2) (1.9238) \right] \]
\[ = 1.9804 + 0.05 \left[ -0.3922 + (-0.2) (3.6983) \right] = 1.9238 \]
\[ y_2^{(2)} = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_2, y_2^{(0)}) \right] = 1.9238 \]
\[ \therefore y_2^{(2)} = y_2^{(3)} = 1.9238 \]
\[ \therefore y_2 = y_2^{(3)} = 1.9238 \]

**Problem 3:** Given \( y' = x + \sin y \), \( y(0) = 1 \), compute \( y(0.2) \) and \( y(0.4) \) with \( h = 0.2 \) using modified Euler's method.

**Solution:** \( x_0 = 0 \); \( y_0 = 1 \); \( h = 0.2 \); \( f(x, y) = x + \sin y \)

By Euler's formula, \( y_1 = y_0 + h f(x_0, y_0) \)
\[ = 1 + 0.2 \left[ 0 + \sin 1 \right] = 1.163 \]
We take it as \( y_1^{(0)} = 1.163 \)

\[ y_1^{(1)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] = 1 + 0.1 \frac{1}{2} \left[ \sin 1 + 1.12 \right] \]
\[ = 1.1961 \]

\[ y_1^{(2)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] = 1 + 0.1 \frac{1}{2} \left[ \sin 1 + 1.1961 \right] \]
\[ = 1.2038 \]
\[ y_1^{(3)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] = 1 + 0.1 \frac{1}{2} \left[ \sin 1 + 1.2038 \right] \]
\[ = 1.2045 \]
\[ y_1^{(4)} = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] = 1 + 0.1 \frac{1}{2} \left[ \sin 1 + 1.2045 \right] \]
\[ = 1.2046 \]
Prob 4: Solve the DE \( \frac{dy}{dx} = 2 + \sqrt{xy} \), \( y(1) = 1 \), by Modified Euler's method and obtain \( y \) at \( x = 2 \) in steps of 0.2.

Soln.: \[ 5.051 \]

Prob 5: Solve numerically \( y' = y + e^x \), \( y(0) = 0 \) for \( x = 0.2, 0.4 \) by Modified Euler's method.

Soln.: \[ 0.24214, 0.59116 \]

Prob 6: Using Modified Euler's method, obtain \( y(0.25) \) given \( y' = 2xy \), \( y(0) = 1 \)

Soln.: \[ 1.0625 \]

Prob 7: Given that \( \frac{dy}{dx} = x^2 + y^2 \), \( y(0) = 1 \), determine \( y(0.1) \) and \( y(0.2) \) using Modified Euler's method.

Soln.: \[ 1.07266, 1.25066 \]

Prob 8: If \( \frac{dy}{dx} = x + 5y \), use Modified Euler's method to approximate \( y \) when \( x = 0.6 \) in steps of 0.2 given \( y(0) = 1 \)

Soln.: \[ 1.8861 \]

Prob 9: Using Modified Euler's method, find an approximate value of \( y \) when \( x = 0.3 \) given \( y' = x + y \), \( y(0) = 1 \)

Soln.: \[ 1.4004 \]

Prob 10: Solve \( y' = x + y \) using Modified Euler's method to approximate \( y \) when \( x = 0.02, 0.04 \) at 0.06 with \( h = 0.02 \)

Soln.: \[ 1.0202, 1.0408, 1.0619 \]
Runge-Kutta Method: (4th order classical method)

In this method, we solve the differential equation

\[ \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \]

So, \[ y_1 = y_0 + \frac{h}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) \]

where \[ k_1 = hf(x_0, y_0) \]

\[ k_2 = hf(x_0 + \frac{h}{2}, y_0 + k_1) \]

\[ k_3 = hf(x_0 + \frac{h}{2}, y_0 + k_2) \]

\[ k_4 = hf(x_0 + h, y_0 + k_3) \]

**Problem 1:** Apply fourth order RK method, to find an approximate value of \( y \) when \( x = 0.2 \) in steps of 0.1, given that \( y' + y = 0 \), \( y(0) = 1 \).

**Soh:** Here \( x_0 = 0, \ y_0 = 0 \); \( h = 0.1 \); \( y' = f(x, y) = -y \)

So, \[ y_1 = y_0 + k \] where \( k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \)

\[ k_1 = hf(x_0, y_0) = (0.1)(-0.1) = -0.01 \]

\[ k_2 = hf(x_0 + \frac{h}{2}, y_0 + k_1) = 0.1 \left[ f(0.05, 0.95) \right] = 0.1 \left[ -0.95 \right] = -0.095 \]

\[ k_3 = hf(x_0 + \frac{h}{2}, y_0 + k_2) = 0.1 \left[ f(0.05, 0.9525) \right] = 0.1 \left[ -0.9525 \right] = -0.09525 \]

\[ k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 \left[ f(0.1, 0.90475) \right] = (0.1)(-0.90475) = -0.090475 \]

So, \[ y_1 = y_0 + \frac{h}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) = 0.9048 \]

Now, \( x_1 = x_0 + h = 0.1 \); \( y_1 = 0.9048 \); \( h = 0.1 \)

\[ y_2 = y(0.2) = y_1 + k \] where \( k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \)

\[ k = hf(x_1, y_1) = 0.1 \left[ -y_1 \right] = 0.1 \left[ -0.9048 \right] = -0.09048 \]
\[ k_2 = h f \left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2} \right) = 0.1 \left( (y_1 + k_1) - 0.1 \right) = 0.085959 \]
\[ k_3 = h f \left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2} \right) = 0.1 \left( (y_1 + k_2) - 0.086185 \right) = -0.086185 \]
\[ k_4 = h f \left(x_1 + h, y_1 + k_3 \right) = 0.1 \left( (y_1 + k_3) - 0.081865 \right) = -0.081865 \]
So \[ y_2 = y_1 + k_2 = y_1 + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) = 0.81873 \]
\[ y(0.2) = 0.81873 \]

**Problem:** Apply 4th order Runge-Kutta method to find \( y(0.1) \) and \( y(0.2) \)
given that \( y' = x^2 + y^2 \), \( y(0) = 1 \);

**John:** Here \( x_0 = 0 \), \( y_0 = 1 \), \( h = 0.1 \), \( f(x, y) = y' = x^2 + y^2 \)
 \[ y_1 = y(0.1) = y_0 + k \] where \( k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \)
\[ k_1 = h f(x_0, y_0) = 0.1 \left( x_0 \cdot y_0 + y_0^2 \right) = 0.1 \left( 0 + 1 \right) = 0.1 \]
\[ k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 \left( \left( x_0 + \frac{h}{2} \right)^2 + \left( y_0 + \frac{k_1}{2} \right)^2 \right) = 0.11655 \]
\[ k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1 \left( \left( x_0 + \frac{h}{2} \right)^2 + \left( y_0 + \frac{k_2}{2} \right)^2 \right) = 0.1122 \]
\[ k_4 = h f(x_0 + h, y_0 + k_3) = 0.1 \left( \left( x_0 + h \right)^2 + \left( y_0 + k_3 \right)^2 \right) = 0.1248 \]
\[ y(0.1) = y_0 + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) = 1 + 0.1133 = 1.1133 \]
\[ y(0.2) = y_1 + k \] where \( k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \)
\[ k_1 = h f(x_0, y_1) = 0.1 \left( x_0 \cdot y_1 + y_1^2 \right) = 0.1351 \]
\[ k_2 = h f(x_0 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1 \left( \left( x_0 + \frac{h}{2} \right)^2 + \left( y_1 + \frac{k_1}{2} \right)^2 \right) = 0.1571 \]
\[ k_3 = h f(x_0 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.1599 \]
\[ k_4 = h f(x_0 + h, y_1 + k_3) = 0.1876 \]
\[
\begin{align*}
\gamma_2 &= y(0.2) = y_1 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) = 1.133 + k \\
&= 1.133 + \frac{1}{6} (0.1351 + 0.3142 + 0.3198 + 0.1876) \\
\therefore y(0.2) &= 1.2728
\end{align*}
\]

**Prob 6.3:** Solve \( y' = x - y \) given that \( y(0) = 0.4 \). Find \( y(1.2) \) using 4th order RK Method.

**Soln:** Here \( x_0 = 1; \ y_0 = 0.4; \ x_1 = 1.1; \ x_2 = 1.2; \ h = 0.1 \)

\[
y_1 = y(1.1) = y_0 + h = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)
\]

\[
k_1 = 0.06, \ k_2 = 0.062, \ k_3 = 0.0619, \ k_4 = 0.06381
\]

\[
y_1 = y_0 + k = 0.4 + \frac{1}{6} (0.06 + 2(0.062) + 2(0.0619) + 0.06381)
\]

\[
y_1 = 0.4619
\]

\[
y_2 = y_1 + k = 0.4619 + \frac{1}{6} (0.7019 + 2(0.03371) + 2(0.067126) + 0.0668)
\]

\[
y_2 = 0.825
\]

**Prob 4:** Solve \( \frac{dy}{dx} = xy \) using 4th order RK method.

for \( x = 0.12 \) given \( y(0) = 1 \), taking \( h = 0.2 \)

**Soln:** \( x_0 = 0; \ y_0 = 1; \ h = 0.2; \ y' = \text{for-}y \) \( \therefore y(0.2) \)

\[
y_1 = y_0 + k = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)
\]

\[
k_1 = 1 + \frac{1}{6} (0 + 2(0.02) + 2(0.0202) + 0.0408)
\]

\[
k_1 = 1.0202
\]

**Prob 6.5:** Using 4th order RK Method solve \( y' = \frac{y-x}{y+x} \)

at \( x = 0.2 \), taking \( h = 0.2 \) with \( y(0) = 1 \).

**Soln:** Here \( x_0 = 0; \ y_0 = 1; \ h = 0.2; \ x_1 = 0.2; \ y' = \text{for} y \)

\[
y_1 = y(0.2) = y_0 + k = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)
\]

\[
= 1 + \frac{1}{6} (0.2 + (0.1666 + 0.16619) + 0.0707) = 1.15667
\]
Predictor- Corrector Methods:

The earlier methods are single step methods. To find $y_{n+1}$, we used only $y_n$. But in these Predictor-Corrector methods, to find $y_{n+1}$, we need not just $y_n$ but also some of the earlier values of $y$ like $y_{n-1}$, $y_{n-2}$, $y_{n-3}$ etc. These methods are called multi-step methods. In this chapter, we discuss two such methods called Milne's method and Adams Bashforth method. They both have similar formulae called Predictor and Corrector.

First we shall discuss Milne's method:

In this method, the predictor is given by

$$y_{n+1}^{(p)} = y_n + \frac{h}{2} \left( 2y_n' - y_{n-1}' + 2y_{n-2}' \right)$$

and the Corrector is given by

$$y_{n+1}^{(c)} = y_{n+1}^{(p)} + \frac{h}{2} \left( y_{n+1}' + 4y_{n+1}' + y_{n+1}' \right)$$

which can be generally written as

$$y_{n+1}^{(p)} = y_n + \frac{h}{2} \left[ 2y_{n-1}' - y_{n-1}' + 2y_{n-1}' \right]$$

$$y_{n+1}^{(c)} = y_{n+1}^{(p)} + \frac{h}{2} \left[ y_{n+1}' + 4y_{n+1}' + y_{n+1}' \right]$$

Problems:

Use Milne's method to find $y(0.8)$ and $y(1.0)$ from $y' = 1 + y$; $y(0) = 0$. Find the initial value $y(0.3)$, $y(0.5)$ and $y(0.6)$ from RK Method.

Solution:

Given $x_0 = 0$, $y_0 = 0$; $h = 0.2$; $x_1 = 0.2$, $y_1 = 1 + y^2$.

$y_1 = y_0 + h = y_0 + \frac{1}{6} (0.2 + 0.404 + 0.40408 + 0.20816)$
\[ y(0.2) = 0.2027 \]

So \( y(1.0) = 0.2027 \)

\[ x_1 = 0.2; \quad y_1 = 0.2027; \quad h = 0.2 \]

\[ y(0.4) = y_2 = y_1 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \]

\[ = 0.2027 + \frac{1}{6} (0.02082 + 2(0.2188) + 2(0.2195) + 0.2356) \]

\[ = 0.4228 \]

\[ x_2 = 0.4; \quad y_2 = 0.4228; \quad h = 0.2 \]

\[ y(0.6) = y_3 = y_2 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \]

\[ y_3 - y(0.6) = 0.4228 + \frac{1}{6} (1.5678) \approx 0.6841 \]

So using Predictor, \( y'_{y_3} = y_0 + y_6 \cdot \left( \frac{2y_1 - y_2 + 2y_3}{3} \right) \)

\[ = 0 + \frac{4}{3} (0.2) \left[ 2(1.0411) - 1.1787 + 2(1.4681) \right] = 1.0239 \]

\[ y_1' = 1 + y_1^2 = 1 + (0.2027)^2 = 1.0411 \]

\[ y_2' = 1 + y_2^2 = 1 + (0.4228)^2 = 1.1787 \]

\[ y_3' = 1 + y_3^2 = 1 + (0.6841)^2 = 1.4681 \]

Now \( y_4' = 1 + y_4^2 = 1 + (1.0239)^2 = 2.0484 \)

The corrector \( y_{y_4} = y_2 + \frac{4}{3} \left( y_2' + 4y_3' + y_4' \right) \)

\[ = 0.4228 + \frac{4}{3} \left( 1.0411 - 1.1787 + 2.0484 \right) \]

\[ = y(0.8) \approx 0.4228 + 0.6066 = 1.0294 \]

To find \( y(1.0) \): The Predictor \( y'_{y_5} = y_4 + \frac{4}{3} \left( 2y_2' - y_3' + 2y_4' \right) \)

\[ y_5' = 1 + y_5^2 = 1 + (0.9944)^2 = 2.05966 \]

\[ y_5 = 0.2027 + \frac{4}{6} (0.2) \left[ 2(1.787) + 1.4681 + 2(2.05966) \right] \]
\[ y(1.0) = 1.5383 \]

The corrector \[ y_5^c = y_3 + \frac{h}{3} (y_1 + 4y_2 + y_5') \]

Where \[ y_5' = 1 + y_5^2 = 1 + (1.5383)^2 = 3.3664 \]

\[ y_5 = y(1.0) = 0.6641 + 0.2 \left[ 1.4681 + 4(2.05796) + 3.3664 \right] = 0.6641 + 0.87154 = 1.5386. \]

**Problem:** Find the slope of \( dy/dx \) at \( x = 0.4 \), \( y(0) = 1 \) using Milne's method. Use Euler's modified method to evaluate \( y(0.1) \), \( y(0.2) \), and \( y(0.3) \).

**Solution:** Here \( x_0 = 0 \), \( x_1 = 0.1 \), \( h = 0.1 \)

By Euler's modified method:

\[ y_1 = y(0.1) = 0.995 \]

\[ y_2 = y(0.2) = 0.8371 \]

\[ y_3 = y(0.3) = 0.7812 \]

Using \( y_1, y_2, y_3 \) we've:

\[ y_1' = y_1 - y_2 = -0.8075 \]

\[ y_2' = y_2 - y_3 = -0.6371 \]

\[ y_3' = y_3 - y_3 = -0.1812 \]

By Milne's predictor:

\[ y_4 = y_0 + \frac{h}{3} (3y_1 + 4y_2 + y_3') \]

\[ = 1 + 0.1 \cdot 0.6371 \cdot \left[ -1.1619 + 0.6371 - 0.1812 \right] = 1 + 0.15508 = 1.15492 \]

\[ y_4' = y_4' = x_4 \cdot y_4 = 0.1 \cdot 1.15492 = 0.14492 \]

\[ y_5 = y_3 + \frac{h}{3} (y_1' + 4y_2' + y_4') = 0.8371 + 0.1 \left[ -0.6371 - 0.7248 - 0.4492 \right] = 0.5383 \]

\[ y(1.0) = 1.5383 \]

\[ \]
\[ y'(0.4) = y_4 = 0.7769 \]

**Problem 63.** Use Milne's method to find \( y(0.3) \) from \( y' = x^2 + y^2 \), \( y(0) = 1 \). Find the initial values \( y(-0.1), y(0.1) \) and \( y(0.2) \) from Taylor's series method.

**Solution:** Here \( h = 0.1 \); \( f(x, y) = x^2 + y^2 \).

- \( y(-0.1) = 0.9087 \);
- \( y(0.1) = 1.1113 \);
- \( y(0.2) = 1.2506 \).

- \( y_0 = 1 \);
- \( y_1 = 1.2449 \);
- \( y_2 = 1.6040 \);

The predictor \( y_3 = y_2 + \frac{h}{3} \left[ 2y_0' - 2y_1' + 2y_2' \right] \)

\[ = 0.9087 + \frac{0.4}{3} (2 - 1.2449 + 2.2080) = 1.4371 \]

- \( y_3 = (0.3)^2 + (1.4371)^2 = 2.1552 \)

The corrector \( y_3^c = y_2 + \frac{h}{3} \left[ 2y_0' - 2y_1' + 2y_2' \right] \)

\[ = 1.113 + \frac{0.4}{3} (1.2449 + 2.4600 + 2.1552) \]

\[ = 1.4385 \]

- \( y_3 = y(0.3) = 1.4385 \).

**Problem 4.** Use Milne's method to solve \( y' = \frac{2y}{x} \) with \( y(1) = 2 \). Compute \( y(2) \) by Runge-Kutta method. Find the startup values using Runge-Kutta method taking \( h = 0.25 \).

**Solution:** \( h = 0.20 \)

**Problem 65.** Use Milne's predictor-corrector method to find the 5th of \( y' + y = \frac{1}{x^2} \) at \( 1.4 \) given \( y(1) = 1 \), \( y(1.1) = 0.996 \), \( y(1.2) = 0.988 \); \( y(1.3) = 0.972 \).

**Solution:** \( y(1.4) = 0.949 \).
Adams-Bashforth Method

In this method, the Predictor is

\[ y_4^{(P)} = y_3 + \frac{h}{24} \left( 55y_3' - 59y_2' + 37y_1' - 9y_0' \right) \]

and the Corrector is

\[ y_4 = y_3 + \frac{h}{24} \left( 9y_4' + 19y_3' - 5y_2' + y_1' \right). \]

If \( y_1, y_2, y_3 \) are obtained using any of the earlier methods like Picard, Taylor, Euler, Modified Euler, or RK Methods, and then evaluate the corresponding derivative \( y_1', y_2', y_3' \), and use them to find the predictor \( y_4^{(P)} \) and using the predictor, we find the corrector.

**Problem:** Apply Adams-Bashforth method and find \( y \) at \( x = 4.4 \) given \( 5xy' + y^2 - 2 = 0 \) & \( y = 1 \) at \( x = 4 \) initially by generating other values using Taylor series expansion etc.

Using the Taylor series expansion, we find

\[ y(4.1) = 1.0049 \] \[ y(4.3) = 1.0142 \]

\[ y(4.2) = 1.0097 \]

Here \( x_0 = 4 \); \( y_0 = 1 \); \( h = 0.1 \).

The Predictor for \( y_4 \), \( y_4^{(P)} = y_3 + \frac{h}{24} \left( 55y_3' - 59y_2' + 37y_1' - 9y_0' \right) \)

Here \( y_0' = \frac{2 - 1}{5.4} = 0.05 \); \( y_1' = 2 - \frac{(1.0049)^2}{5 \times 4.1} = 0.0483 \)

\[ y_2' = \frac{2 - (1.0097)^2}{5 \times 4.2} = 0.0467 \]

\[ y_3' = \frac{2 - (1.0142)^2}{5 \times 4.3} = 0.0452 \]

So \( y_4^{(P)} = 1.0142 + \frac{0.1}{24} \left[ 55 \left( 0.0452 - 59(0.0467) + 37(0.0483) - 9(0.05) \right) \right] = 1.0187 \)
The corrector is \( y_4^{(c)} = y_3 + \frac{h}{2u} \left[ 9y_4' + 17y_3' - 5y_2' + y_1' \right] \).

\[
= 1.042 + \frac{0.1}{2u} \left[ 9 \left( 0.0187 \right) + 17 \left( 0.0452 \right) - 5 \left( 0.0467 \right) + 0.0485 \right]
\]

\[
= 1.0186 \cdot \frac{2 - 0.0187^2}{5 \cdot (4.4)}
\]

\[
y_4 = y_4(4.4) = 1.0186.
\]

Prove: Solve \( y' + y + xy^2 = 0 \) with \( y_0 = 1; \ y_1 = 0.9008; \ y_2 = 0.8066 \)

\( y_3 = 0.722 \) w.r.t \( x_0 = 0; \ x_1 = 0.1; \ x_2 = 0.2; \ x_3 = 0.3 \) respectively.

Find \( y \) when \( x = 0.4 \) using Adams-Bashforth method.

Here

\[
\begin{align*}
\text{at } x_0 &= 0 & \text{at } x_1 &= 0.1 & \text{at } x_2 &= 0.2 & \text{at } x_3 &= 0.3 \\
y_0 &= 1 & y_1 &= 0.9008 & y_2 &= 0.8066 & y_3 &= 0.722
\end{align*}
\]

Now the predictor \( y_4^{(p)} = y_3 + \frac{h}{2u} \left[ 5.5y_3' - 5y_2' + 37y_1' - 9y_0' \right] \)

\[
\begin{align*}
\text{Here } y_0' &= - (y_0 + x_0 y_0^2) = -1 \\
y_1' &= - (y_1 + x_1 y_1^2) = -0.9819 \\
y_2' &= - (y_2 + x_2 y_2^2) = -0.9367 \\
y_3' &= - (y_3 + x_3 y_3^2) = -0.8784
\end{align*}
\]

\[
y_4^{(p)} = 0.722 + \frac{0.1}{2u} \left[ 5.5 \left( -0.9819 \right) - 5 \left( -0.9367 \right) + 37 \left( -0.8784 \right) - 9 \left( -1 \right) \right]
\]

\[
y_4^{(P)} = 0.6371
\]

\[
y_4' = - (y_4 + x_4 y_4^2) = -0.7795
\]

Now the corrector \( y_4^{(c)} = y_4 + \frac{h}{2u} \left( 9y_4' + 17y_3' - 5y_2' + y_1' \right) \)
\[ y_4 = 0.6379 \]

Write this \( \eta^{(4)}_4 \), \( y_4' = -0.6379 + (0.4)(0.6379)^2 = -0.8007 \)

Using this again in the predictor we get

\[ y_4^{(1)} = 0.722 + \frac{0.1}{24} \left[ 9(-0.8007) + 19(-0.8784) - 5(-0.9367) - 9(-1) \right] \]
\[ = 0.6379 \]

\( \Rightarrow y_4 = y(0.4) = 0.6379 \).

**Problem 3.** Using Adams Bashforth method find \( y(1.4) \)
given \( y' = \left(2y + \frac{dy}{dx}\right)/2 \), given \( y(0) = 2 \). Find \( y(1.1), y(1.2), y(1.3) \) using Taylor Series expansion of order 4.

\( \text{Sol.} \quad 0.0723 \)

**Problem 4:** Using Adams Bashforth method find \( y(1.4) \)
given \( y' + \left(\frac{dy}{dx}\right) = \frac{1}{x} \), given \( y(1) = 1; y(1.1) = 0.996 \)
\( y(1.2) = 0.9766; y(1.3) = 0.972 \).

\( \text{Sol.} \quad 0.949 \)

**Problem 5:** Use Taylor Series expansion of order 4 to find \( y(0.1), y(0.2), y(0.3) \) ad then solve \( y' + y = x^2; y(0) = 1 \) using

Adams-Bashforth method & Milne's method

\( \text{Sol.} \quad 0.6897 \).