Picard's method:

In this method we consider the system

\[ \frac{dy}{dx} = f_1(x, y, z) \]
\[ \frac{dz}{dx} = f_2(x, y, z) \]

with given initial conditions \( y(x_0) = y_0 \) and \( z(x_0) = z_0 \).

So by Picard's method, known to us we can write:

\[ y = y_0 + \int_{x_0}^{x} f_1(x, y, z) \, dx \]
\[ z = z_0 + \int_{x_0}^{x} f_2(x, y, z) \, dx \]

Using known values of \( y \) and \( z \) on RHS we can write:

\[ y = y_0 + \int_{x_0}^{x} f_1(x, y_0, z_0) \, dx \]
\[ z = z_0 + \int_{x_0}^{x} f_2(x, y_0, z_0) \, dx \]

We write them as \( y_1 \) and \( z_1 \) respectively.

We write:

\[ y_1 = y_0 + \int_{x_0}^{x} f_1(x, y_1, z_1) \, dx \]
\[ z_1 = z_0 + \int_{x_0}^{x} f_2(x, y_0, z_0) \, dx \]

Further, \( y_2, y_3 \ldots \) are similarly obtained.

Problem: Find second approximation to the solution of

\[ x' = (t+1)/2 \quad y' = 2t/3z \quad z(x_0) = 2/3 \quad y(0) = 3 \]

Using Picard's method.

Take \( x_0 = 1 \):

\[ y_1 = 3 + \int_{x_0}^{x} \left( t^2 - \frac{4}{3} \right) \, dx \]

\[ = 3 + \left[ \frac{t^3}{3} - \frac{4}{3} \right]_{1/3} = \frac{5}{2} + \frac{4}{3} \]
\[ x_2 = \frac{2}{3} + \frac{1}{2} \int_1^t \left( \frac{1}{2} + y_1 \right) \, dt \quad \text{and} \quad y_2 = 3 + \int_1^t \frac{t \cdot x_1}{3x_1} \, dt \]

\[ = \frac{2}{3} + \frac{1}{2} \left[ \frac{7t}{2} + \frac{t^3}{6} - \frac{7}{6} \right] \quad \text{and} \quad = 3 + \frac{1}{3} \int_1^t \left( t + \frac{2}{3t-2} \right) \, dt \]

\[ = \frac{7t}{4} + \frac{t^3}{6} - \frac{7}{12} \quad \text{and} \quad = 3 + \frac{1}{3} \left[ t + \frac{2}{3} \left( \log(3t-2) \right) \right] \]

\[ = \frac{8}{3} + \frac{t}{3} + \frac{2}{9} \log(3t-2) \]

The second approx. are

\[ x_2 = \frac{t^3}{12} + \frac{7t}{4} - \frac{7}{6} \quad \text{and} \quad y_2 = \frac{t}{3} + \frac{2}{9} \log(3t-2) + \frac{8}{3} \]

**Problem:** Using Picard's method of successive approximation, find \( y \) & \( z \) at \( x = 0.1 \) given \( \frac{dy}{dx} = x \) and \( \frac{dz}{dx} = x^3(y+z) \), \( y(0) = 1 \) and \( z(0) = 0.5 \).

**Solution:** Here \( x_0 = 0 \); \( y_0 = 1 \); \( z_0 = 0.5 \); \( y' = f(x, y, z) = x \); \( z' = g(x, y, z) = x^3(y+z) \). Hence, we have

\[ y_1 = y_0 + \int_{x_0}^{x} f(x, y, z_0) \, dx \quad \text{and} \quad z_1 = z_0 + \int_{x_0}^{x} g(x, y_0, z_0) \, dx \]

\[ = 1 + \int_{0}^{x} \frac{1}{2} \, dx \quad \text{and} \quad = \frac{1}{2} + \int_{0}^{x} \frac{3}{2} \, dx \]

\[ = 1 + \frac{x}{2} \quad \text{and} \quad = \frac{1}{2} + \frac{3x}{2} \]

\[ y_2 = 1 + \int_{0}^{x} \left( \frac{1}{2} + \frac{3x}{5} \right) \, dx \quad \text{and} \quad z_2 = \frac{1}{2} + \int_{0}^{x} \left( \frac{3}{2} + \frac{x}{2} + \frac{3x}{6} \right) \, dx \]

\[ = 1 + \frac{x}{2} + \frac{3x}{40} \quad \text{and} \quad = \frac{1}{2} + \frac{3x}{8} + \frac{x}{10} + \frac{3x}{64} \]

Taking \( x = 0.1 \) on the RHS of these two expressions for \( y_2 \) & \( z_2 \), we get \( y(0.1) = 1.05 \) & \( z(0.1) = 0.50 \).
Problem 3: Solve \( \frac{dy}{dx} = x + z \) \& \( \frac{dz}{dx} = x - y^2 \) given that \( y(0) = 2 \) \& \( z(0) = 1 \).

Using Picard's method of successive approximations, order 2.

Ans: \( y(0.1) = 0.0645 \) \& \( z(0.1) = 0.5867 \).

Problem 4: Using Picard's method, solve the system of equations

\[
\begin{align*}
\frac{dy}{dx} &= y + 2x; \\
\frac{dz}{dx} &= 3y + 2z \quad \text{given} \quad y(0) = 6; \quad z(0) = 4 \\
\end{align*}
\]

by finding three approximations and hence find \( y \) \& \( x \) at \( x = 0.1, 0.2 \).

Ans: \( y_3 = 6 + 14x + 33x^2 + \frac{127}{3}x^3; \quad z_3 = 4 + 26x + 47x^2 + \frac{193}{3}x^3; \)

\( y(0.2) = 6.2935; \quad z(0.2) = 4.5393 \).

Runge-Kutta 4th order method: In this method, we consider the two simultaneous equations

\[
\frac{dy}{dx} = f_1(x, y, z) \quad \& \quad \frac{dz}{dx} = f_2(x, y, z) \quad \text{with the initial conditions} \quad y(x_0) = y_0 \quad \& \quad z(x_0) = z_0.
\]

We evaluate \( y(x_1) = y(x_0 + h) \) \& \( z(x_1) = z(x_0 + h) \) using

\[
\begin{align*}
k_1 &= h f_1(x_0, y_0, z_0) \quad \& \quad l_1 = h f_2(x_0, y_0, z_0) \\
k_2 &= h f_1(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) \quad \& \quad l_2 = h f_2(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) \\
k_3 &= h f_1(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) \quad \& \quad l_3 = h f_2(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) \\
k_4 &= h f_1(x_0 + h, y_0 + k_3, z_0 + l_3) \quad \& \quad l_4 = h f_2(x_0 + h, y_0 + k_3, z_0 + l_3)
\end{align*}
\]

Then \( y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \) \& \( z(x_1) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \).

Similar procedure is adopted to find \( y(x_2) \) \& \( z(x_2) \) etc.
Problem 1: Apply 4th order RK Method to solve \( \frac{dy}{dx} = y + z \) and \( \frac{dz}{dx} = -y + z \), given \( y(0) = 0 \) and \( z(0) = 0 \) at \( x = 0 \). Take \( h = 0.1 \).

Given:

- \( x_0 = 0 \), \( y_0 = 0 \), \( z_0 = 0 \), \( h = 0.1 \)
- \( f(x, y, z) = y + z \), \( g(x, y, z) = -y + z \)

So:

- \( k_1 = hf(x_0, y_0, z_0) = 0.1 \)
- \( k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{k_1}{2}) = 0.115 \)
- \( k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{k_2}{2}) = 0.1105 \)
- \( k_4 = hf(x_0 + h, y_0 + k_3, z_0 + k_3) = 0.112 \)

\[ y_1 = y(0.1) = y_0 + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) = 0.11033 \]

\[ z_1 = z(0.1) = z_0 + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) = 0.0976 \]

Problem 2: Using 4th order RK Method, solve the 1st order equation \( \frac{dy}{dt} = 2x + y \), given \( y(0) = 0 \) and \( x(0) = 0 \) at \( t = 0 \). Find \( y(0.2) \).

Given:

- \( x_0 = 0 \), \( y_0 = 0 \) at \( t = 0 \)

So:

- \( k_1 = hf(t_0, x_0, y_0) = 0.1 \)
- \( k_2 = hf(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{k_1}{2}) = 0.09 \)
- \( k_3 = hf(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{k_2}{2}) = 0.078 \)
- \( k_4 = hf(t_0 + h, x_0 + k_3, y_0 + k_3) = 0.093 \)

\[ y(0.2) = y_0 + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) = 0.1 + \frac{0.2}{6} (0.1 + 2(0.09) + 2(0.078) + 0.093) = 0.188 \]
\[ x_1 = x(0.1) = x(0.12) = x_{0} + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) = 0.0948 \]
\[ y_1 = y(0.1) = y_{0.12} = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) = 0.2287 \]
Numerical solutions of 2nd order DEs: 

**Picard's Method:** In this method, the given 2nd order DE is reduced to two first order simultaneous differential equations and then apply Picard's method to solve simultaneous DEs.

**Problem:** Solve the 2nd order DE $y'' + y'y' = x^3$ given that $y(1) = 1$ & $y'(1) = 1$ at $x = 1.1$.

**Solution:** Take $y' = dy/dx = x$ then we get $y'' = d/dx (dy/dx) = d^2x/dx$;

So in given 2nd order DE becomes $d^2x/dx + y^2x = x^3; \ y(1) = 1$ \ $x(1) = 1$.

Thus the two 1st order simultaneous eqns are:

\[ \frac{dy}{dx} = x \text{ and } \frac{dz}{dx} = x^3 - y^2 ; \ y(1) = 1 \text{ and } x(1) = 1 \]

\[ x_0 = 1; \ y_0 = 1; \ z_0 = 1 \]

\[ y_1 = 1 + \int_1^x z \ dx \]

\[ y_2 = 1 + \int_1^x \left( x^3 - y_1^2 \right) \ dx \]

\[ y_1 = 1 + \left( x - \frac{x^2}{2} \right) \]

\[ y_2 = 1 + \left( \frac{x^4}{4} - \frac{x^6}{6} \right) \]

\[ y_2 = \frac{-3}{10} + \frac{7x}{4} - \frac{3x^3}{2} + \frac{x^5}{20} \]

$z_2$ is not required as $z_2$ is not required as $y_2$ value is only $y$ value is required.
Problem: When a pendulum swings in a resisting medium, its equation of motion is of the form \( \frac{d^2 \theta}{dt^2} + a \frac{d \theta}{dt} + b \sin \theta = 0 \)

where \( a \) & \( b \) are constants. Let \( a = 0.2 \) & \( b = 10 \). Find the second approximation to the solution of the equation with \( \theta(0) = 0.3 \) when \( t = 0.01 \), using Picard's method. Find \( \theta \) & \( \phi \) at \( t = 0.02 \) & \( t = 0.03 \).

Solve: Given \( \theta(0) = 0.3 \) & \( \frac{d \theta}{dt} \) \( t = 0 \) \( \frac{d \theta}{dt} = 0 \) & \( \frac{d^2 \theta}{dt^2} + 0.2 \frac{d \theta}{dt} + 10 \sin \theta = 0 \)

put \( \phi = \frac{d \theta}{dt} \Rightarrow \frac{d \phi}{dt} = \frac{d^2 \theta}{dt^2} \); so the 2nd order DE becomes:

\( \frac{d \phi}{dt} + 0.2 \phi + 10 \sin \theta = 0 \)

where \( \phi(0) = 0 \); \( \theta(0) = 0.3 \); \( t = 0 \);

so we've the system of equations:

\( \frac{d \theta}{dt} = \phi \) & \( \frac{d \phi}{dt} = -0.2 \phi + 10 \sin \theta \)

so we get:

\( \int \frac{d \phi}{\phi} = -\int (0.2 \phi + 10 \sin \theta) \; dt \)

\( \int_{0.3}^{\theta} \phi \; dt = \int_{0}^{t} (0.2 \phi + 10 \sin \theta) \; dt \)

\( \Rightarrow \theta = 0.3 + \int_{0}^{t} \phi \; dt \)

\( \Rightarrow \theta = 0.3 + \int_{0}^{t} (0.2 \phi + 10 \sin \theta) \; dt \)

\( \Rightarrow \phi_1 = -2.9552 t \)

\( \phi_2 = 0.3 + \int_{0}^{t} \phi_1 \; dt \)

\( = 0.3 - \int_{0}^{t} 2.9552 \; dt \)

\( = 0.3 - 2.9552 \frac{t^2}{2} \)

\( \Rightarrow \theta_2 = 0.3 - 1.4776 t^2 \)

\( \Rightarrow \theta(0.01) = 0.29985 \); \( \theta(0.02) = 0.29941 \) & \( \theta(0.03) = 0.29867 \)
Problem 3: By 4th order RK method, find y(0.1) given that

\[ \frac{dy}{dx} = y^3 \text{ and } y = 10, \frac{dy}{dx} = 5 \text{ at } x = 0 \]

Sols: - Put \( \frac{dy}{dx} = z \Rightarrow \frac{dz}{dx} = \frac{dz}{dx} \). So that we've

\[ \frac{dy}{dx} = z & \quad \text{and} \quad \frac{dz}{dx} = y^3 \text{ where } x = 0; \quad y(x_0) = 10; \quad z(x_0) = 5 \]

\[ f(x, y, z) = z; \quad g(x, y, z) = y^3; \quad h = 0.1 \]

\[ k_1 = h f(x_0, y_0, z_0) = 0.1 \cdot 5 = 0.5 \]

\[ k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{h}{2}) = 5.5 \]

\[ k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{h}{2}) = 5.885 \]

\[ k_4 = h f(x_0 + h, y_0 + k_3, z_0 + h) = 21.227 \]

\[ y_1 = y(x_0 + h) = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) = 17.4162 \]

Problem 4: Given \( y'' - xy' - y = 0; \quad y(0) = 1; \quad y'(0) = 0 \). Find \( y(0.2) \) and \( y'(0.2) \) using RK 4th order method.

Sols: - Take \( y' = \frac{dy}{dx} = x; \Rightarrow \frac{dy}{dx} = y'' = \frac{dz}{dx} \).

\[ \text{we have } \frac{dy}{dx} = z; \quad \frac{dz}{dx} = xx + y \text{ at } x = 0; \quad y_0 = 1; \quad z_0 = 0; \quad h = 0.2 \]

\[ k_1 = h f(x_0, y_0, z_0) = 0 \]

\[ k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{h}{2}) = 0.02 \]

\[ k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{h}{2}) = 0.202 \]

\[ k_4 = h f(x_0 + h, y_0 + k_3, z_0 + h) = 0.408 \]

\[ y_1 = y(x_0 + h) = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) = 1 + 0.0202 = 1.0202 \]

\[ x_1 = x(x_0 + h) = x_0 + 0.2 = 0.2 (x_0 = 0) \]

\[ y_1 = y(0.2) = y(0.2) \]

\[ x_1 = x(0.2) = 0.2 \]
Probs. - Using 4th order RK Method find the values of x at
\[ \frac{dy}{dx} \text{ at } t=0.1 \text{ given } \frac{d^2y}{dt^2} + t \frac{dy}{dt} + 4y = 0 \text{ and } y(0)=3, y'(0) = 0 \]
at x = 0;
Solution: \[ \frac{dy}{dt} = y; \quad \frac{dx}{dt} = ty - 4y \]
x0 = 0; y0 = 3; x0 = 0; h = 0.1
k1 = 0; k2 = -0.106; k3 = -0.06615; k4 = -0.1191; \nu_1 = 1, \nu_2 = -1.203
l3 = -1.191; l4 = -1.118785; y(x0+h) = 2.7401; \frac{dy}{dx} \text{ at } x = 1.196.

Problem - Solve \[ \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - 6y = 0 \text{ using Picard's method given } \]
x = 0 and y = 1, x = 0.1;
Solution: \[ \frac{dy}{dx} = x; \quad \frac{d^2y}{dx^2} = 3x - 6y \]
x0 = 0; y0 = 1; x0 = 0.1; h = 0.1
y1 = 1 + \frac{x}{10}; y2 = 1 + \frac{x}{10} + \frac{3x^2 - 20y}{20}; y3 = 1 + \frac{x}{10} + \frac{3x^2 - 20y + 9x^3}{400}
\nu_1 = 1 + \frac{6x - 3x^2}{20}; \nu_2 = 1 + 6x + \frac{3x^2 - 6x^3 + 9x^4}{80}; \nu_3 \text{ is not required.}

Milne's method to solve 2nd order ODE:

Q1: Find the value of y(x0+\nu) given \[ \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \text{ at } x = 0 \]
y = y0 = 0 at x = 0; y(0.1) = 0.01; y0(0.1) = 0.1996
y(0.4) = 0.0795; y1(0.4) = 0.3937; y(0.6) = 0.1762; y0(0.6) = 0.5684
Use the corrector formula twice to obtain the y0 value of x = 0.8
Solution: Here \[ \frac{dy}{dx} = 2; \quad \frac{d^2y}{dx^2} = -1 - 2y \]
when x0 = 0; y0 = 0; x0 = 0; \nu = 1.
when x = 0.2; y1 = 0.02; x1 = 0.1996; \nu_1 = 0.992.
when x = 0.4; y2 = 0.0795; x2 = 0.3937; x2 = 0.9374.
when x = 0.6; y3 = 0.1762; x3 = 0.5684; \nu_3 = 0.7975.
\[
\begin{align*}
    y_4 &= y_0 + \frac{h}{3} (2z_2 - z_1 + 2z_3), \\
    z_4 &= z_0 + \frac{h}{3} (2z_1' - z_2' + 2z_3') \\
    y_{41} &= 0.3049, \quad z_{41} = 0.7055
\end{align*}
\]

Using the corrector:
\[
\begin{align*}
    y_4^{(c)} &= y_2 + \frac{h}{3} (z_2 + z_3 + z_4) \\
    z_4^{(c)} &= z_2 + \frac{h}{3} (z_2' + z_3' + z_4')
\end{align*}
\]

Let us \( z'^1 = 1 - 2y_4^{(p)} \) \( z''_4 = 1 - 2(0.3049)(0.7055) = 0.569 \) \( y_4^{(c)} = 0.3045 \) \( z_4^{(c)} = 0.7074 \) again using the corrector formula:
\[
\begin{align*}
    y_4^{(c)} &= 0.0795 + \frac{h}{3} \left[ 0.3045 + 4(0.569) + 0.7074 \right] = 0.3046 \\
    y(0.8) &= 0.3066, \quad z(0.8) \text{ is not required.}
\end{align*}
\]

Problem: Find a solution of \( y'' + xy' + y = 0 \), \( y(1) = 1; y'(1) = 0 \).
First apply Picard’s method to find the 3rd approx. Then use Hlhe’s method to find \( y(0.4) \).

Solution: Take \( \frac{dy}{dx} = z \) \( \Rightarrow \frac{d^2z}{dx^2} = -x \frac{dy}{dx} - y \), \( y_0 = 0; z_0 = 1; z_0 = 0 \)
\[
\begin{align*}
    y &= \int \frac{dy}{dx} dx = \int x \ dx = \frac{x^2}{2} + C_1 \quad \Rightarrow C_1 = 0 \\
    z &= \int (xy + y) dx = \int x^2 dx = \frac{x^3}{3} + C_2 \quad \Rightarrow C_2 = 0
\end{align*}
\]

\[
\begin{align*}
    y_1 &= 1 + \int_0^x z \ dx \quad \Rightarrow z_1 = -\int_0^x x \ dx = -\frac{x^2}{2} \\
    y_2 &= 1 + \int_0^x z_1 \ dx \quad \Rightarrow z_2 = -x + \frac{x^3}{2} \\
    y_3 &= 1 + \int_0^x (z_2 + \frac{x^3}{3}) \ dx \quad \Rightarrow z_3 = -x + \frac{x^3 - \frac{2x^5}{5}}{15}
\end{align*}
\]

Taking \( a = 0.1, \) 0.2 and 0.3 we get:
\[
\begin{align*}
    y(0.1) &= 0.995, \quad y(0.2) = 0.9801, \quad y(0.3) = 0.956
\end{align*}
\]
\[ z(0.1) = -0.0985; z(0.2) = -0.196; z(0.3) = -0.2867. \]

\[ \text{Also} \quad z'(0) = -(0 + 1) = -1; \quad z'(0.1) = -\left[ (0.1)(-0.0985) + 0.985 \right] = -0.985; \]
\[ z'(0.2) = -\left[ (0.2)(-0.196) + 0.980 \right] = -0.941; \quad z'(0.3) = -\left[ (0.3)(-0.2867) + 0.980 \right] = -0.875. \]

**Now**

When \( x_1 = 0 \), \( y_1 = 1 \), \( z_1 = 1 \), \( z_0 = -1 \)

When \( x_1 = 0.1 \), \( y_1 = 0.985 \), \( z_1 = -0.0985 \), \( z_1' = -0.985 \)

When \( x_2 = 0.2 \), \( y_2 = 0.9801 \), \( z_2 = -0.196 \), \( z_2' = -0.941 \)

When \( x_3 = 0.3 \), \( y_3 = 0.9867 \), \( z_3 = -0.2867 \), \( z_3' = -0.875 \).

So by Milne's predictor:

\[ y_4^{(p)} = y_0 + \frac{h}{3} \left[ 2z_1' - z_2 + 2z_3 \right] = 0.9231; \]
\[ z_4^{(p)} = y_0 + \frac{h}{3} \left[ 2z_1' - z_2' + 2z_3' \right] = -0.3692; \]
\[ \Rightarrow z_4' = -(x_4 z_4^{(p)} + y_4^{(p)}) = -0.7754. \]

**Now using the correctors we get**

\[ y_4 = y_0 + \frac{h}{3} \left[ 2z_2' + 2z_3' + z_4' \right] = 0.9230; \]
\[ z_4 = z_2' + \frac{h}{3} \left[ 2z_1' + 4z_3' + z_4' \right] = -0.3692; \]
\[ z_4' = 0.9230. \]

**Proof:** Solve the DE \( y'' = 2y' \), \( y = 0.8 \) using Milne's Predictor Corrector formula.

- Given corrector formula:
  - \( y(0.1) = 0 \);
  - \( y(0.2) = 0.2027 \);
  - \( y(0.3) = 0.4228 \);
  - \( y(0.4) = 0.6841 \);
  - \( y(0.5) = 1 \);
  - \( y(0.6) = 1.468 \).

- We use corrector formula since \( x = 0.8 \).

The equation is:

\[ \frac{dy}{dx} = z; \quad \frac{dz}{dx} = 2yz. \]

**Solution**

\[ z(0.1) = 0; \quad z'(0.2) = 0.422; \quad z'(0.4) = 0.997; \quad z'(0.6) = 2.009; \]
\[ y_4 = 1.0237; \quad z_4 = 2.0307; \quad y_4 = 1.0285; \quad z_4 = 2.0564. \]
\[ z_4' = 4.1157; \quad y_4' = 1.03009. \]