
Engineering Mathematics – IV

Time: 3 hrs. Max. Marks: 100

Note: 1. Answer FIVE full questions, selecting at least TWO questions from each part.
2. Use of statistical tables is permitted.

PART – A

1. a. Using Taylor series method, solve the problem \( \frac{dy}{dx} = x^2y - 1 \), \( y(0) = 1 \) at the point \( x = 0.2 \).

   Consider upto 4\(^{th}\) degree terms. (06 Marks)

b. Using R.K. method of order 4, solve \( \frac{dy}{dx} = 3x + \frac{y}{2} \), \( y(0) = 1 \) at the points \( x = 0.1 \) and \( x = 0.2 \) by taking step length \( h = 0.1 \). (07 Marks)

c. Given that \( \frac{dy}{dx} = x - y \), \( y(0) = 0 \), \( y(0.2) = 0.02 \), \( y(0.4) = 0.0795 \), \( y(0.6) = 0.1762 \). Compute \( y \) at \( x = 0.8 \) by Adams-Bashforth predictor-corrector method. Use the corrector formula twice. (07 Marks)

2. a. Evaluate \( y \) and \( z \) at \( x = 0.1 \) from the Picards second approximation to the solution of the following system of equations given by \( y = 1 \) and \( z = 0.5 \) at \( x = 0 \) initially.

   \( \frac{dy}{dx} = z \), \( \frac{dz}{dx} = x(y + z) \) (06 Marks)

b. Given \( y'' - xy' - y = 0 \) with the initial conditions \( y(0) = 1 \), \( y'(0) = 0 \). Compute \( y(0.2) \) and \( y'(0.2) \) by taking \( h = 0.2 \) and using fourth order Runge-Kutta method. (07 Marks)

c. Applying Milne’s method compute \( y(0.8) \). Given that \( y \) satisfies the equation \( y'' = 2yy' \) and \( y \) and \( y' \) are governed by the following values. \( y(0) = 0 \), \( y(0.2) = 0.2027 \), \( y(0.4) = 0.4228 \), \( y(0.6) = 0.6841 \), \( y(0.8) = 1 \), \( y(0.2) = 1.041 \), \( y'(0.4) = 1.179 \), \( y'(0.6) = 1.468 \). (Apply corrector only once). (07 Marks)

3. a. Derive Cauchy Riemann equations in Cartesian form. (06 Marks)

b. Find an analytic function \( f(z) = u + iv \). Given \( u = x^2 - y^2 + \frac{x}{x^2 + y^2} \). (07 Marks)

c. If \( f(z) \) is a regular function of \( z \), show that \( \left| \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right| \left| f(z) \right|^2 = 4 \left| f'(z) \right|^2 \). (07 Marks)

4. a. Find the bilinear transformation that maps the points \( z = 1, i, -i \) onto the points \( w = 1, i, -1 \) respectively. (06 Marks)

b. Find the region in the w-plane bounded by the lines \( x = 1, y = 1, x + y = 1 \) under the transformation \( w = z^2 \). Indicate the region with sketches. (07 Marks)

c. Evaluate \( \int_{C} \frac{e^{2z}}{(z + 1)(z - 2)} \, dz \) where \( C \) is the circle \( |z| = 3 \). (07 Marks)
5 a. Solve the Laplace equation in cylindrical polar coordinate system leading to Bessel differential equation. (06 Marks)

b. If \( \alpha \) and \( \beta \) are two distinct roots of \( J_n(x) = 0 \) then prove that \( \int_0^1 x J_n(\alpha x) J_n(\beta x) \, dx = 0 \) if \( \alpha \neq \beta \). (07 Marks)

c. Express the polynomial, \( 2x^3 - x^2 - 3x + 2 \) in terms of Legendre polynomials. (07 Marks)

6 a. State and prove addition theorem of probability. (06 Marks)

b. Three students A, B, C write an entrance examination. Their chances of passing are \( \frac{1}{3} \), \( \frac{1}{4} \), \( \frac{1}{4} \) respectively. Find the probability that,
   i) At least one of them passes.
   ii) All of them pass.
   iii) At least two of them pass. (07 Marks)

c. Three machines A, B, C produce respectively 60%, 30%, 10% of the total number of items of a factory. The percentages of defective outputs of these three machines are respectively 2%, 3% and 4%. An item is selected at random and is found to be defective. Find the probability that the item was produced by machine C. (07 Marks)

7 a. The pdf of a random variable \( x \) is given by the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(x) )</td>
<td>k</td>
<td>2k</td>
<td>3k</td>
<td>4k</td>
<td>3k</td>
<td>2k</td>
<td>k</td>
</tr>
</tbody>
</table>

Find:
   i) The value of \( k \)
   ii) \( P(x > 1) \)
   iii) \( P(-1 < x \leq 2) \)
   iv) Mean of \( x \) (06 Marks)

v) Standard deviation of \( x \).

b. In a certain factory turning out razor blades there is a small probability of 1/500 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing, i) One defective, ii) Two defective, in a consignment of 10000 packets. (07 Marks)

c. In a normal distribution 31% of items are under 45 and 8% of items are over 64. Find the mean and standard deviation of the distribution. (07 Marks)

8 a. A sample of 100 tyres is taken from a lot. The mean life of tyres is found to be 39350 kilometers with a standard deviation of 3260. Can it be considered as a true random sample from a population with mean life of 40000 kilometers? (Use 0.05 level of significance) Establish 99% confidence limits within which the mean life of tyres expected to lie. (Given that \( Z_{0.05} = 1.96 \), \( Z_{0.01} = 2.58 \)) (06 Marks)

b. Ten individuals are chosen at random from a population and their heights in inches are found to be 63, 63, 66, 67, 68, 69, 70, 71, 71. Test the hypothesis that the mean height of the universe is 66 inches. (Given that \( t_{0.05} = 2.262 \) for 9 d.f.) (07 Marks)

c. Fit a Poisson distribution to the following data and test the goodness of fit at 5% level of significance. Given that \( \chi^2_{0.05} = 7.815 \) for 4 degrees of freedom.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>122</td>
</tr>
<tr>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

* * * * *
1. a. 
\[ y' = x^2y - 1, \quad y(0) = 1, \quad y'(0) = -1 \]
\[ y'' = x^2y' + y' + 2x, \quad y''(0) = 0 \]
\[ y''' = x^2y'' + y'y' + 2x, \quad y'''(0) = 2 \]
\[ = x^2y'' + 4xy' + 2y \]
\[ y'' = 2y'' + 4y'y' + 2xy'' + x^2y''' \]
\[ = 6y'y' + 2xy'' + x^2y''' \]
\[ y''(0) = -6 \]

The Taylor's series expansion is given by:
\[ y(x) = y_0 + (x-x_0)y'_0 + (x-x_0)^2 \frac{y''_0}{2} + (x-x_0)^3 \frac{y'''_0}{3} + \ldots \]
\[ = 1 + x(-1) + x^2 \frac{1}{2} (0) + \frac{x^3}{6} 2 + \frac{x^4}{4} (-6) \]
\[ y(0.2) = 0.8023 \]

b. 
\[ y' = 3x + \frac{y}{2}, \quad y(0) = 1 \]
\[ k_1 = h f(x_0, y_0) = 0.1 f(0,1) = 0.05 \]
\[ k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f[(0.05), 1.025] \]
\[ = 0.1 \left[ 3 (0.05) + \frac{1.025}{2} \right] \]
\[ = 0.06625 \]
\[ k_3 = \frac{h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})}{2} = 0.01 f[(0.05), 1.0331] \]
\[ = 0.01 \left[ 3 (0.05) + 1.0331 \right] \]
\[ K_3 = 0.0666, \quad K_4 = h \cdot f(x_0 + h, y_0 + k_3) \]
\[ z_{0.1} f(0.1, 1.0666) = 0.0823 \]

\[ y(0.1) = y_0 + \frac{1}{6} \left[ k_1 + 2k_2 + 2k_3 + k_4 \right] = 1.0665 \]

**2nd stage**
\[ x_0 = 0.1, \quad y_0 = 1.0665 \]
\[ k_1 = 0.0833, \quad k_2 = 0.1004, \quad k_3 = 0.10084, \quad k_4 = 0.1183 \]
\[ y(0.2) = y_0 + \frac{1}{6} \left[ k_1 + 2k_2 + 2k_3 + k_4 \right] = 1.16473 \]

\[ y' = x - y^2 \]
\[ x \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \]
\[ y \quad y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \]
\[ y' \quad 0 \quad 0 \quad 0.1996 \quad 0.3937 \quad 0.5689 \]

\[ y_4(r) = y_3 + \frac{h}{24} \left[ 55y_3 - 59y_2 + 37y_1 - 9y_0 \right] \]
\[ = 0.1762 + 0.2 \frac{1}{24} \left[ 55(0.5689) - 59(0.3937) + 37(0.1996) - 9(0) \right] \]
\[ = 0.3049 \]

\[ y_4' = x_4 - \left[ y_4(r) \right]^2 = 0.7071 \]

\[ y_4(c) = y_3 + \frac{h}{24} \left[ 9y_4' + 19y_3 - 5y_2 + y_1 \right] \]
\[ = 0.1763 + 0.2 \frac{1}{24} \left[ 9(0.7071) + 19(0.5689) - 5(0.3937) + (0) \right] \]
2. a. \( y' = z \), \( z' = x^3(y+z) \), \( y_0 = 1, z_0 = 0.5 \). 

\[ y = 1 + \int_0^x z \, dx \]

Put \( z = \frac{1}{2} \) in (1), \( y = 1, z = \frac{1}{2} \) in (2)

**1st approx.**

\[ y_1 = 1 + \int_0^x \left( \frac{1}{2} \right) \, dx \]

\[ z_1 = \frac{1}{2} + \int_0^x \frac{3}{2} x^2 \, dx \]

\[ y_1 = 1 + \frac{x}{2} \]

\[ z_1 = \frac{1}{2} + \frac{3x^3}{8} \]

**2nd approx.**

\[ y_2 = 1 + \int_0^x z_1 \, dx \]

\[ z_2 = \frac{1}{2} + \int_0^x \frac{3x^3(y_1+z_1)}{2} \, dx \]

\[ y_2 = 1 + \frac{x}{2} + \frac{3x^5}{40} \]

\[ z_2 = \frac{1}{2} + \frac{3x^9 + x^5 + 3x^8}{8} \]

\( y(0.1) = 1.05 \), \( z(0.1) = 0.5 \)

b. \( y' = z \), \( y'' - x y' - y = 0 \), \( y_0 = 1, z_0 = 0 \)

\( y' = z \), \( z' = x \) \( z + y \)

\( k_1 = h f(x_0, y_0, z_0) = 0.2 \)

\( f(0.1, 0) = 0.2 \)

\( \Delta t = h \) \( g(x_0, y_0, z_0) = 0.2 \)
\[ k_2 = h f \left[ x_0 + h, y_0 + \frac{k_1}{2}, z_0 + \frac{h}{2} \right] = 0.02 \]
\[ l_2 = h g \left[ x_0 + h, y_0 + \frac{k_1}{2}, z_0 + \frac{h}{2} \right] = 0.202 \]
\[ k_3 = h f \left[ x_0 + h, y_0 + \frac{k_2}{2}, z_0 + \frac{h}{2} \right] = 0.20 \]

\[ k_4 = h f \left[ x_0 + h, y_0 + k_3, z_0 + h \right] = 0.2 \]

\[ l_4 = h g \left[ x_0 + h, y_0 + k_3, z_0 + h \right] = 0.212 \]

\[ y(0.2) = 1.0202, \quad y'(0.2) = 0.204 \]

\begin{align*}
\text{c.} & \quad y' = z, \quad y'' = 2yy' \\
& \quad x_0 = 0, \quad y_0 = 0, \quad z_0 = 1
\end{align*}

\begin{align*}
& \quad x = 0, \quad 0.2, \quad 0.4, \quad 0.6 \\
& \quad y = 0, \quad 0.2027, \quad 0.4228, \quad 0.6841 \\
& \quad y' = z = 1, \quad 1.041, \quad 1.174, \quad 1.468 \\
& \quad z = 2yz = 0, \quad 0.422, \quad 0.997, \quad 2.009
\end{align*}

\[ y_4^{(p)} = y_0 + \frac{4h}{3} \left[ 2z_1 - z_2 + 2z_3 \right] \]
\[ z = 0 + 4 \left( \frac{0.2}{3} \right) \left[ 2(1.041) - 1.174 + 2(1.468) \right] \]
\[ = 1.0227 \]

\[ z_4^{(p)} = z_0 + \frac{4h}{3} \left[ a_2z_1' - z_2' + 2z_3' \right] \]
\[ = 0 + 4 \left( \frac{0.2}{3} \right) \left[ 2(0.422) - 0.997 + 2(2.009) \right] \]
\[ = 2.0307 \]

\[ z_4^{(p)} - z_4^{(p)} = 4.15 \]
$y_4^{(c)} = y_2 + \frac{1}{3} \left[ z_2 + 4z_3 + z_4 \right] = 1.0282$

$z_4^{(c)} = z_2 + \frac{1}{3} \left[ z_2' + 4z_3' + z_4' \right] = 9.0584$

$z_4' = 4.1577, \quad y(0.8) = 1.0301$

3.

Cauchy – Riemann Equations in Cartesian form.

Statement: The necessary conditions that the function $w = f(z) = u(x, y) + iv(x, y)$ may be analytic at any point $z = x + iy$ is that if four continuous first order partial derivatives $u_x, u_y, v_x, v_y$ satisfy the equations $u_x = v_y$ and $v_x = -u_y$. These are known as C-R equations.

Proof: Let $f(z)$ be analytic at a point $z = x + iy$ and hence by the definition $f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$ exists and is unique.

In the Cartesian form

$f(z) = u(x, y) + iv(x, y)$ and let $\Delta z$ be the increment in $z$ corresponding to the increments $\Delta x, \Delta y$ in $x, y$. 

\[ f'(z) = \lim_{\delta z \to 0} \left( \frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - u(x, y) - iv(x, y)}{\delta z} \right) \]

\[ f'(z) = \lim_{\delta z \to 0} \left( \frac{u(x+\delta x, y+\delta y) - u(x, y)}{\delta z} + \frac{iv(x+\delta x, y+\delta y) - iv(x, y)}{\delta z} \right) \]

\[ \delta z = \delta x + i\delta y \]

\[ \text{as } \delta z \to 0, \text{ we have 2 possibilities} \]

\text{Case 1: Let } \delta y = 0 \Rightarrow \delta z = \delta x \]

\[ \text{as } \delta z \to 0 \Rightarrow \delta x \to 0 \]

\[ 0 \text{ becomes} \]

\[ f'(z) = \lim_{\delta x \to 0} \left( \frac{u(x+\delta x, y) - u(x, y)}{\delta x} + i \lim_{\delta y \to 0} \left( \frac{v(x+\delta x, y) - v(x, y)}{\delta y} \right) \right) \]

\[ f'(z) = \frac{u_x}{\delta x} + iv_x \quad \text{(By basic def of partial derivative)} \]

\text{Case 2: Let } \delta x = 0 \Rightarrow \delta z = i\delta y \]

\[ \text{as } \delta z \to 0 \Rightarrow i\delta y \to 0 \quad \text{ie } \delta y \to 0 \]

\[ \text{Now } 0 \text{ becomes} \]

\[ f'(z) = \lim_{\delta y \to 0} \left( \frac{u(x, y+\delta y) - u(x, y)}{\delta y} \right) \]
Let \( v(x, y + \delta y) - v(x, y) \)
\[ \delta y \to 0 \]

\( f'(z) = -iuy + vy \) (By basic diff)

(3) Partial derivatives

Comparing eqs 2 & 3 we get:

\[ u_x = vy \quad \& \quad v_x = -uy \]. These are C-Reqs

\[ f'(z) = \frac{ux + ivy}{(x^2 + y^2)^2} \]

\[ f'(z) = \frac{u_x - iuy}{(x^2 + y^2)^2} \quad \text{(By C-Reqs)} \]

\[ = 2x + \frac{y^2 - x^2}{(x^2 + y^2)^2} - i \left( \frac{-2x - 2xy}{(x^2 + y^2)^2} \right) \]

Put \( x = 3, y = 0 \)

\[ = 2 \frac{3}{3} + \frac{-3^2}{(3^2)^2} - i \left( \frac{-0}{(3^2)^2} \right) \]

\[ = 2 \frac{3}{3} - \frac{1}{3^2} \]

\[ \text{on int} \quad f(z) = \frac{3^2}{3} + i\frac{1}{3} + c \]

C. Let \( f(z) = u + iv \) be analytic
\[ f(z) = \sqrt{u^2 + v^2} \]  
\[ f(z) = u + iv \]

To prove \( \phi_{xx} + \phi_{yy} = 4 |f(z)|^2 \)

\[ \phi = u^2 + v^2, \quad \phi_x = 2uv, \quad \phi_y = 2uv \]

\[ \phi_{xx} = 2(u_{xx} + u_x^2 + v_{xx} + v_x^2) \]  
\[ \phi_{yy} = 2(u_{yy} + u_y^2 + v_{yy} + v_y^2) \]

Adding (1) and (2) we get

\[ \phi_{xx} + \phi_{yy} = 2 \left[ (u_{xx} + u_x^2 + v_{xx} + v_x^2) + (u_{yy} + u_y^2 + v_{yy} + v_y^2) \right] \]

As \( u \) and \( v \) are harmonic \( u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0 \). By CRR, \( u_x = v_y \) and \( v_x = -u_y \)

\[ \phi_{xx} + \phi_{yy} = 2 \left[ (u_x^2 + v_x^2) + (u_y^2 + v_y^2) \right] \]

\[ = 2 \left[ 2 (u_x^2 + v_x^2) \right] \]

\[ = 4 (u_x^2 + v_x^2) \]

\[ |f'(z)|^2 = \sqrt{u_x^2 + v_x^2} \]

\[ 1 \cdot \phi_{xx} + \phi_{yy} = 4 |f(z)|^2 \]

4. \( z = -1, i \) \( -1, w = 1, i, -1 \)

\[ w = \frac{az + b}{cz + d} \quad \text{ad - bc} \neq 0 \]
\[
\frac{(w - w_4)}{(w_3 - w_3)(w_2 - w_1)} = \frac{(z - z_2)(z_3 - z_3)}{(z_3 - z_3)(z_2 - z_1)}
\]

\[
\frac{(w - 1)(w + 1)}{(w + 1)(w - 1)} = \frac{(z + 1)(z + 1)}{(z + 1)(z + 1)}
\]

\[
(w - 1) = i(w + 1) \Rightarrow w(1 - i) = (i + 1)
\]

\[
(w - 1) = i(w + 1)
\]

\[
2m
\]

b. \( w = z^2 \)

\[
\nu + \nu = (x + iy)^2 = (x^2 - y^2) + 2ixy
\]

\[
u = x^2 - y^2 \quad \nu = 2xy
\]

Consider \( x = 1 \), Eq becomes \( \nu = 1 - y^2 \), \( \nu = 2y \)

Substitute \( \nu = y \) in \( \nu \) we have \( \nu = 1 - \nu^2 \)

\( \nu^2 = 4(1 - \nu) \). This is parabola in \( u \)-plane with vertex \( (1, 0) \) and symmetrical about the \( u \)-axis.

Consider \( y = 1 \). Eq becomes \( \nu = x^2 - 1, \nu = 2x \)

Sub \( \nu = x \) in \( \nu \) we have \( \nu = \nu^2 - 1 \)

\( \nu^2 = 4(1 + \nu) \). This is also a parabola in the \( u \)-plane with vertex \( (-1, 0) \) and symmetrical about the \( u \)-axis.

Consider \( x + y = 1 \) or \( y = 1 - x \).
1. becomes \( u = x^2 - (1-x)^2 \)

\[ u = -1 + 2x \rightarrow u = 2x(1-x) \]

Substituting \( 2x = 1 + u \), \( x = \frac{1}{2}(1 + u) \), we have

\[ v = (1 + u)(1 - \frac{1 + u}{2}) = (1 + u)(1 - u) \quad \text{eq} \]

\[ v = \frac{1}{2}(1 - u^2) \quad \text{eq} \]

\( 1 + u^2 = 2v \), \( u^2 = -2 [v - \frac{1}{2}] \). This is also a parabola in the \( w \)-plane with vertex \((0, \frac{1}{2})\) symmetrical about the \( v \)-axis.

\[ \text{Z-plane} \]

\[ \text{V-plane} \]

\[ \text{C.} \quad I = \int \frac{e^{2z}}{c} \frac{dz}{(z+1)(z-2)} \quad |z| = 3 \]

\[ \text{Sol.} \quad \text{The points} \quad z = a = -1, \quad z = a = 2 \quad \text{being} \]

\((-1,0)\) \((2,0)\) \text{lies inside} \quad |z| = 3

\[ \frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2} \quad \text{eq} \]

Resolving into partial fractions:

\[ A = -\frac{1}{3}, \quad B = \frac{1}{3} \quad \text{eq} \]

\[ \frac{e^{2z}}{(z+1)(z-2)} = -\frac{1}{3} \frac{1}{z+1} + \frac{1}{3} \frac{1}{z-2} \]
\[
\int \frac{e^{2z}}{(z+1)(z-2)} \, dz = \frac{1}{3} \left[ \int \frac{e^{2z}}{z-2} \, dz - \int \frac{e^{2z}}{z+1} \, dz \right] \\
\int \frac{f(z)}{z-a} \, dz = 2\pi i \, f(a) \quad \text{(By Cauchy's integral formula)}
\]

\[
\int \frac{e^{2z}}{(z+1)(z-2)} \, dz = \frac{1}{3} \left[ 2\pi i \, e^4 - \frac{2\pi i}{e^2} \right] \\
\int \frac{e^{2z}}{(z+1)(z-2)} \, dz = \frac{2\pi i}{3} \left[ e^4 - \frac{1}{e^2} \right]
\]

5. a. The coordinates \((r, \phi, z)\) are called cylindrical coordinates, and the relationship with the Cartesian coordinates \((x, y, z)\) is given by \(x = r \cos \phi\), \(y = r \sin \phi\), \(z = z\).

The Laplace eq \(\nabla^2 f = 0\) in the cylindrical system is given by

\[
\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 \quad (1)
\]

We shall solve this by the method of separation of variables.

Let \(f = f_1(r) f_2(\phi) f_3(z)\) be the solution of (1) where

\(f_1 = f_1(\phi), \quad f_2 = f_2(\phi), \quad f_3 = f_3(z)\)

\(\therefore (1)\) becomes
\[ \frac{d^2 f_1}{d s^2} + \frac{1}{5} \frac{d (H_2 F_3)}{d s} + \frac{1}{5} \frac{d^2 H_2 F_3}{d f_2} + \frac{2}{5} \frac{d H_2 F_3}{d f_2} = 0 \]

\[ \frac{1}{f_1} \frac{d^2 f_1}{d s^2} + \frac{1}{f_1} \frac{d f_1}{d s} + \frac{1}{f_1} \frac{d^2 f_2}{d f_2} + \frac{1}{f_1} \frac{d^2 f_3}{d f_2} = -1 \frac{d f_3}{d z^2} \quad \text{(2)} \]

Let us set \( \frac{d^2 f_3}{d z^2} = 1 \)

\( \text{(2)} \) becomes

\[ \frac{1}{f_1} \frac{d^2 f_1}{d s^2} + \frac{1}{f_1} \frac{d f_1}{d s} + \frac{1}{f_1} \frac{d^2 f_2}{d f_2} = -1 \]

Now x'y' by \( s^2 \) we get

\[ \frac{s^2}{f_1} \frac{d^2 f_1}{d s^2} + \frac{s}{f_1} \frac{d f_1}{d s} + \frac{1}{f_1} \frac{d^2 f_2}{d f_2} = -s^2 \]

\[ \frac{s^2}{f_1} \frac{d^2 f_1}{d s^2} + \frac{s}{f_1} \frac{d f_1}{d s} + s^2 = -1 \frac{d^2 f_2}{d f_2} \]

Again L.H.S is a function of \( s \) & R.H.S is a function of \( f_2 \):

\[ \frac{s^2}{f_1} \frac{d^2 f_1}{d s^2} + \frac{s}{f_1} \frac{d f_1}{d s} + s^2 = n^2 \]
\[ \frac{P^2}{P} \frac{d^2 f}{dp^2} + \frac{P}{P} \frac{df}{dp} + (P^2 - n^2) = 0 \]

This eq can be written in the form

\[ x^2 y'' + xy' + (x^2 - n^2)y = 0. \]

This is the Bessel's differential eq of order \( n \).

b. If \( x \) \& \( B \) are 2 distinct roots of \( J_n(x) = 0 \)
then \( \int \frac{x}{J_n(x)} J_n(Bx) \ dx = 0 \). If \( x \neq B \)

**Proof:** \( J_n (\lambda x) \) is a solution of the eq

\[ x^2 y'' + xy' + (x^2 - n^2)y = 0 \]

If \( u = J_n(\lambda x) \) \& \( v = J_n(Bx) \) the associated differential eqs are

\[ x^2 u'' + xu' + (x^2 - n^2)u = 0 \quad (1) \]

\[ x^2 v'' + xv' + (B^2 x^2 - n^2)v = 0 \quad (2) \]

\[ x^2 y \text{ by } \frac{u}{x} \quad \text{\&} \quad y \text{ by } \frac{v}{x} \]

\[ n v u'' + v u' + x^2 u x - n^2 u \]
\[ n v u'' + v u' + x^2 u x - n^2 \frac{u}{x} = 0 \]
On subtracting we obtain
\[ x (v u'' - w v'') + (v u' - w u') + (x^2 - \beta^2) u w x = 0 \]
\[ \frac{d}{dx} \left( x (v u' - w u') \right) \bigg|_{x=0}^{x=1} = (\beta^2 - x^2) \int_0^1 x u v d x \]
\[ (v u' - w u') \bigg|_{x=0}^{x=1} = 0 \Rightarrow (\beta^2 - x^2) \int_0^1 x u v d x. \]

\[ u = J_0(\beta x), \quad v = J_0(\alpha x) \text{ we have } u' = \beta J_0'(\beta x) \quad \text{and as a consequence of (5) becomes} \]
\[ \int_0^1 [J_0(\beta x) J_0'(...)] - J_0(\alpha x) \beta J_0'(\beta x) \int_0^1 x = 1 \]
\[ (\beta^2 - x^2) \int_0^1 x J_0(\alpha x) J_0(\beta x) d x. \]

Hence \[ \int_0^1 x J_0(\alpha x) J_0(\beta x) d x = \frac{1}{\beta - \alpha} J_0(\beta) J_0'(\alpha). \]
\[ \beta \neq \alpha \] and \[ \beta \neq \alpha \] are distinct roots of \( J_0(x) = 0 \) we have \( J_0(\alpha x) = 0 \) \( \Rightarrow \) \( J_0(\beta x) = 0 \) with the result the R.H.S of (5) becomes zero provided \( \beta^2 - x^2 \neq 0 \) or \( x \neq \beta \).

Thus we have proved that if \( x \neq \beta \)
\[ \int_0^1 x J_0(\alpha x) J_0(\beta x) d x = 0. \]

C. \[ 2x^3 - x^2 - 3x + 2. \]
\[ x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) \]

\[ x = P_1(x) \]

\[ a^2 x^3 - x^2 - 3x + 2 = 2 \left( \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) \right) - \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x) - 3 P_1(x) + 2 P_0(x) \]

\[ = \frac{4}{5} P_3(x) + \frac{6}{5} P_1(x) - \frac{1}{3} P_0(x) - \frac{2}{3} P_2(x) - 3 P_1(x) + 2 P_0(x) \]

\[ = \frac{4}{5} P_3(x) - \frac{2}{3} P_2(x) + P_1(x) - \frac{9}{3} + \frac{5}{3} P_0(x) \]

6.

a. Addition theorem of probability

The probability of the happening of one or the other mutually exclusive events is equal to the sum of the probabilities of the two events, i.e., if \( A, B \) are mutually exclusive events then

\[ P(A \text{ or } B) = P(A) + P(B) \]

Proof: Let the total number of exhaustive, mutually exclusive and equally possible cases in the trials be \( n \). Out of these
m, Cases be favourable to the event A
by m2 cases be favourable to B.

Hence the no of cases favourable to either A or B is m + m2

\[ P(A \text{ or } B) = \frac{m + m2}{n} = \frac{m1 + m2}{n} \]  \[12m\]

m, cases are favourable to A, \(P(A) = \frac{m1}{n}\)
m2 cases are favourable to B, \(P(B) = \frac{m2}{n}\)

Substituting in R.H.S of \[1\]

\[ P(A \text{ or } B) = P(A) + P(B) \]  \[2m\]

b.

\[ P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{3}, \quad P(C) = \frac{1}{4} \]

\[ P(\overline{A}) = \frac{1}{2}, \quad P(\overline{B}) = \frac{2}{3}, \quad P(\overline{C}) = \frac{3}{4} \]  \[2m\]

i) atleast one of them Passes

\[ P(A \cup B \cup C) = 1 - [P(\overline{A}) \cdot P(\overline{B}) \cdot P(\overline{C})] \]

\[ = 1 - \left( \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \right) = \frac{3}{4} \]  \[1m\]

ii) All of them Passes

\[ P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \]

\[ = \frac{1}{24} \]  \[1m\]

iii) Atleast two of them Passes
P(A) \cdot P(B) \cdot P(C) + P(A) \cdot P(B) \cdot P(C) + P(A) \cdot P(B) \cdot P(C)

\begin{align*}
\Rightarrow & \quad \frac{7}{24} \\
6c. & \quad P(A) = 0.6, \quad P(B) = 0.3, \quad P(C) = 0.1
\end{align*}

Let \( E \) be the event of defective output from 3 machines.

\begin{align*}
P(E|A) = 0.02 & \quad P(E|B) = 0.03 \\
P(E|C) = 0.04
\end{align*}

To find \( P(C|E) \)

By Bayes' theorem for conditional Prob we have

\begin{align*}
P(C|E) &= \frac{P(C) \cdot P(E|C)}{P(A) \cdot P(E|A) + P(B) \cdot P(E|B) + P(C) \cdot P(E|C)} \\
\Rightarrow & \quad 0.16
\end{align*}

i) The value of \( k \) is \( P(x) = 1 \)

\begin{align*}
k + 2k + 3k + 4k + 3k + 2k + k &= 1 \\
k &= \frac{1}{16}
\end{align*}
b. \[ P(x > 1) = P(2) + P(3) = \frac{2}{16} + \frac{1}{16} = \frac{3}{16} \]

b. Mean of \( x \)

\[ \mu = \sum x \cdot P(x) = 0 \]

b. Variance

\[ \sigma^2 = \sum (x - \mu)^2 \cdot P(x) \]

\[ = \frac{5}{16} \]

b. Standard Deviation

\[ \sigma = \sqrt{\sigma^2} = \sqrt{\frac{5}{16}} = 1.5811 \]

b. Let \( x \) be the number of defective parts

\[ P(x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} \]

i) One defective

\[ 10,000 \cdot \binom{10,000}{1} \cdot 0.02^1 \cdot 0.98^9 = 196 \]

ii) Two defective

\[ 10,000 \cdot \binom{10,000}{2} \cdot 0.02^2 \cdot 0.98^9 = 22 \]

c. Let \( \mu \) and \( \sigma \) be the mean and S.D. of the normal distribution

\[ P(x < 35) = 0.02 \]

\[ P(x < 60) = 0.89 \]

S.D. \( \sigma = \frac{x - \mu}{6} \)

When \( x = 35 \), \( Z = \frac{35 - \mu}{6} = 2.1 \)
When \( x = 60 \) \( \frac{z = 60 - \mu}{\sigma} = z_2 \)

\[ P(z < z_1) = 0.07, \quad P(z < z_2) = 0.89 \]
\[ 0.5 + \Phi(z) = 0.07, \quad 0.5 + \Phi(z) = 0.89 \]
\[ \Phi(z_1) = 0.43, \quad \Phi(z_2) = 0.39 \]

\[ \Phi(z_1) = -\Phi(1.4757), \quad \Phi(z_2) = \Phi(1.2263) \]

\[ z_1 = -1.4757, \quad z_2 = 1.2263 \]

\[ \frac{35 - \mu}{\sigma} = -1.4757 \]

\[ \frac{60 - \mu}{\sigma} = 1.2263 \]

\[ \mu - 1.4757 \sigma = 35, \quad \mu + 1.2263 \sigma = 60 \]

By solving \( \mu = 48.65, \quad \sigma = 9.25 \)

8.

a. Assume the null hypothesis

\( H_0: \mu = 40,000 \) & alternate hypothesis

\( H_1: \mu \neq 40,000 \)

\( n = 39,350, \quad \bar{x} = 40,000, \quad \sigma = 3260 \)

\( n = 100 \)

\[ z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = 1.9447 \]

\( z = 1.9447, \quad z_{0.05} = 1.96 \)

Ho is rejected i.e. we cannot say that it is a true sample from a population with mean 40,000. Now 99% confidence}
The limits within which the population mean is expected to lie is given as $\bar{x} \pm 2.58 \frac{s}{\sqrt{n}} = (3.8509, 4.1911)$

b. $\bar{x} = \frac{\sum x}{n} = 67.8$, $M = 66$, $S^2 = \sum (x_i - \bar{x})^2$  

$\frac{t}{\sqrt{n}} = 21.89 < t_{0.05} = 2.261$  

The hypothesis is accepted at 5% level of significance

$c. \bar{x} = \frac{\sum x}{n} = 0.5$  

We take this as the mean of the Poisson distribution $\lambda = 0.5$

Hence, expected frequency are given by $E_i = N e^{-\lambda} \frac{\lambda^x}{x!}$

$N = 200$, $x = 0, 1, 2, 3, 4$

$0$: 122, 60, 15, 2, 1

$E_i$: 121, 61, 15, 3, 6

$\chi^2 = \frac{(O_i - E_i)^2}{E_i} = 0.025$  

This is less than $\chi^2_{0.05} = 7.815$

Hence, fitness is considered good.